Optimization of Structures Based on Fracture Mechanics and Reliability Criteria

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Spacecraft structural systems and subsystems are subjected to a number of qualification tests in which the proof loads are chosen at some level above the simulated loads expected during the space mission. Assuming fracture as prime failure mechanism, and allowing for time effects due to cyclic and sustained loadings, this paper treats an optimization method in which the statistical variability of loads and material properties are taken into account, and in which the proof load level is used as an additional design variable. In the optimization process, the structural weight is the objective function whereas the total expected cost due to coupon testing for material characterization, caused by failure during proof testing, and due to mission degradation is a constraint. Numerical results indicate that for a given expected cost constraint, substantial weight savings and improvements of reliability can be realized by proof testing.

Nomenclature

V_j = jth volume element
a_j = flaw size in V_j
R_j = critical resisting stress of V_j
S_j = applied stress normal to the plane of a_j
x_μ, x_0, k = parameters of Weibull distribution
v = unit volume
R_{ju} = uniaxial tensile strength of V_j
R = initial resisting strength of the structure
S = applied load to the structure
ϕ_{j1}, ϕ_{j2} = function of spatial coordinates such that $S\phi_{j1}$ and $S\phi_{j2}$ are analyzed principal stresses at V_j
ϕ_1, ϕ_2 = function of spatial coordinates
V_c = coupon volume
R_c = uniaxial tensile strength of coupon specimen
$W(n)$ = function representing cycles to failure
$U(t)$ = function representing time to failure
R_0 = structural resisting strength after passing proof test
$R(n)$ = structural resisting strength after n loading cycles with amplitude S_c
$R(T)$ = structural resisting strength after the application of S_s for a period of time T
S_0 = proof load
S_s = amplitude of cyclic loading
S_c = magnitude of sustained loading
$F_{R_0}(x)$ = distribution function of R_0
$F_R(x)$ = distribution function of R
p_0 = probability of structural failure under proof load S_0
p_c = probability of structural failure under n cycles of S_c, after surviving proof load S_0
p_{cs} = probability of structural failure under sustained loading S_s for a period of time T, after surviving cyclic loading S_c
p_s = probability of structural failure under sustained loading S_s for a period of time T, after surviving proof load S_0
EC = expected cost of the entire system
EC_i = expected cost of the ith structural subsystem
ϵ_i = S_{0i}/R_i = proof testing level
C_i = cost of coupon tests for the ith subsystem
q_i = expected number of the ith subsystem failing under proof load S_{0i}
C_{0i} = cost of loosing one ith subsystem under proof load S_{0i}
p_{fi} = probability of failure of the ith subsystem during the mission
C_f = cost of mission degradation
S_{0i} = proof load for the ith subsystem
R_i = resisting strength of the ith subsystem
S_i = applied stress to the ith subsystem
$\bar{R}_i$ = mean value of R_i
$\bar{S}_i$ = mean value of S_i
ν_i = $\bar{R}_i/\bar{S}_i$ = central safety factor of the ith subsystem
p_{0i} = probability of failure of the ith subsystem under proof load S_{0i}
α_i = C_{0i}/C_f
B_i = constant parameter indicating the rate of increase of coupon test cost
β_i = B_i/C_f
EC^* = EC/C_f
EC_i^* = EC_i/C_f
EC_a^* = maximum relative expected cost constraint
G_i = weight of the ith subsystem
G = weight of the entire system
h = thickness of the pressure vessel
$\bar{R}_c$ = mean value of R_c
σ_{Rc} = coefficient of variation of R_c
$\bar{R}$ = mean resisting strength of the entire system
ϵ_i^* = optimum proof testing level for the ith subsystem

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I. Introduction

SPACECRAFT structural systems and subsystems are subjected to a number of qualification tests, where the structural components have to withstand specified environments (proof loads) which are intended to simulate the various types of induced stresses at some level above those expected during the space mission. The purpose of these proof tests is to eliminate those structures or structural components which are too weak.

It is clear that after the structural system or subsystem has passed the proof load tests, its statistical strength characteristics have radically changed. For instance, assuming negligible time effects during proof stress application, the statistical strength distribution, at the particular point under consideration in the structure after the test, will be truncated at the lower end up to the proof stress level. A similar statement holds for the proof load applied to a structure as a whole and the resulting truncated over-all structural strength dis-

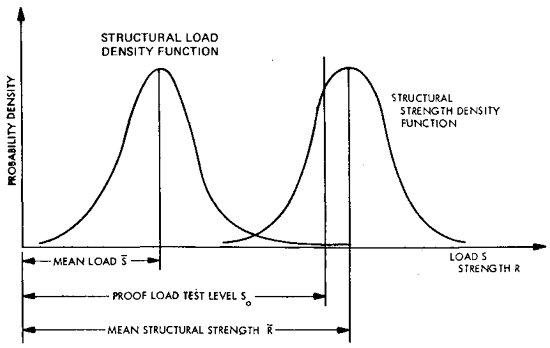

Fig. 1 Typical load and strength distributions.

tribution. This truncation eliminates, through the proof test, precisely that portion of the strength distribution which is least known and which has the greatest interaction with the applied load distribution (see Fig. 1). Reference 1 explores some implications of screening out weak elements with a proof test, when the proof stress distributions are similar, dissimilar and identical to the applied stress distributions and it studies also some simple aspects of optimum designs.

Reference 2 introduces the proof load test level as a design parameter. It is shown that the choice of the proof load test level considerably influences the reliability of the structure, and that in the associated optimization problem, with weight as the objective function and total expected cost as constraint, an optimum proof load test level can usually be obtained with an increase of reliability and a decrease in weight as compared to conventional optimum designs where the proof load has not been considered in the reliability evaluation. In that investigation, the expected cost includes the expected cost of mission failure due to structural failure plus the expected cost of failed subsystems (or components) during proof load testing.

Frequently the optimum proof load test level is considerably different from the mean structural strength. Since the probability density function of the structural strength is usually only known with certain degree of confidence close to the mean strength, it would be necessary, in such cases, to establish, through extensive specimen testing, the strength density functions at points far apart from their mean values. This can add substantial additional costs to the over-all evaluation process if this information is not available.

In Ref. 3, this additional cost item has been allowed for by assuming a linear increase of testing cost with an increase of the difference between mean strength and proof test level. The effect of this additional cost is an increasing tendency to reduce the difference between mean strength and proof load test level as this additional testing cost is increased.

In this paper, which includes the essential aspects of Ref. 4, a similar approach as in Ref. 3 is taken in the optimization process by also including in the expected cost constraint, the cost of establishing the truncated strength distribution function by specimen testing. Fracture mechanics dealing with essentially brittle fracture has been chosen here as an area of application, since fracture mechanics design, as compared to other structural design approaches, where considerable bulk yield occurs before failure, tends to be more susceptible to statistical strength variations. The wide scatter of the results of tests leading to failure is assumed to be an inherent property of the material and is treated as such for the purpose of the present paper which intends to investigate trends of behavior rather than absolutes. A detailed discussion of various probability models for the characterization of material properties and their implications with respect to specimen size has been given in Ref. 5. In addition, the present paper also includes volume effects and allows for time effects, such as fatigue due to cyclic loading and flaw growth due to sustained loading.

In spacecraft structural design, fracture mechanics concepts can be most readily and meaningfully applied to pressure vessel subsystems. In this paper, detailed consideration, by example, is given to pressure vessel design optimization, although the concepts put forth are equally applicable in other areas of design. As will become apparent from this investigation, this approach tends to become more meaningful as the systems, subsystems and components become less expensive relative to the over-all project cost.

II. Statistical Aspects of Fracture

Fracture mechanics treat failures that occur because of the presence of existing, or during loading induced, flaws in the material. The stress intensifications under load application which occur at the edges of a flaw correspondingly increase locally the stored energy. If this stored energy just ahead of the flaw tip becomes larger than the energy needed to create the new surfaces resulting from an extension of the flaw tip, then the flaw increases until energy balance has been restored, see Refs. 6–8. If the applied stress is high enough, or of additional energy in some other form is applied, the energy balance at the flaw tip will not be restored and catastrophic failure will occur. This phenomena is, in principle, common to all fracture failures, although the stress levels causing fracture depend on many parameters describing flaw location, flaw shape, environmental characteristics, loading history, and so forth.[9]

Let the structure be subdivided into small material volume elements, each containing one flaw. The size of the flaw in the jth volume element V_j is described by a parameter a_j which is related to the resisting stress R_j and the applied stress S_j normal to the plane of the flaw by

$$R_j = A_c a_j^{-1/2}; \quad S_j = A a_j^{-1/2} \tag{1}$$

where A_c is a constant describing parametric conditions at the critical state $S_j = R_j$, and A describes these conditions for $S_j < R_j$.

Whereas, for given conditions, A_c is reasonably constant, the flaw sizes, and hence the resistances R_j, typically exhibit wide statistical variations. Thus, knowing the statistical characteristics of the flaw sizes, the statistical properties of the corresponding resistances can be computed using Eq. (1). However, present state of technology does not allow, except in trivial cases, the direct measurement of the statistical distribution of the flaw sizes. In this paper the statistical distributions of the resisting stresses are determined from results of uniaxial tensile tests of specimens, henceforth referred to as coupon tests. The statistical properties and the results of such coupon tests were investigated extensively.[10–12]

Without essential loss of generality, a two-dimensional stress field (plane stress), as associated with thin-walled structures commonly used in space applications, is assumed in the following.

Let R and S be, respectively, the resisting strength of the structure and the applied load, and let $S\phi_{j1}$ and $S\phi_{j2}$ be the analyzed principal stresses at V_j due to the application of S, where ϕ_{j1} and ϕ_{j2} are functions of the spatial coordinates and stiffness properties defining the structure. It is assumed that the strengths of the volume elements are statistically independent and identically distributed and that the angular orientations of the flaws are uniformly distributed between 0 and $\pi/2$. It is further assumed that the mean angular value $\pi/4$ can be used as an approximation. The following experimentally verified Weibull distribution; Refs. 1, 5, and 10–12, can be used for the uniaxial tensile strength R_c of the coupon specimen with volume V_c:

$$F_{R_c}(x) = 1 - \exp\{-(V_c/v)[(x - x_\mu)/x_0]^k\}; \quad x \geq x_\mu \tag{2}$$

where x_μ, x_0 and k are parameters characterizing the particular material, and v is the unit volume. The estimation of x_u, x_0

and k in Eq. 2 can be obtained from the results of coupon tests, for instance, by the method of moments using the standard procedures.

The distribution function of the uniaxial tensile strength R_{ju} of the jth volume element follows from Eq. (2), Refs. 1, 5 and 10–12,

$$F_{R_{ju}}(x) = 1 - \exp\{-(V_j/v)[(x - x_\mu)/x_0]^k\}; \quad x \geq x_\mu \tag{3}$$

Hence, the distribution of the structural resistance R can be shown, Ref. 9, as

$$F_R(S) = 1 - \prod_{j=1} \exp\left\{-\frac{V_j}{v}\left[\frac{S(\phi_{j1} + \phi_{j2}) - x_\mu}{x_0}\right]^k\right\} \tag{4a}$$
$$S(\phi_{j1} + \phi_{j2}) \geq x_\mu$$

with the approximation

$$F_R(S) = 1 - \exp\left\{-\int_V \frac{1}{v}\left[\frac{S(\phi_1 + \phi_2) - x_\mu}{x_0}\right]^k dv\right\} \tag{4b}$$
$$S(\phi_1 + \phi_2) \geq x_\mu$$

For a spherical pressure vessel, as example, in which approximately $\phi_1 = \phi_2 = \phi =$ constant, the distribution function of the vessel strength becomes

$$F_R(S) = 1 - \exp\left\{-\frac{V}{v}\left(\frac{2S\phi - x_\mu}{x_0}\right)^k\right\}; \quad 2S\phi \geq x_\mu \tag{5}$$

in which V is the total material volume of the pressure vessel.

It follows from Eqs. (4) and (5) that the distribution function of the strength of the entire structure is also a Weibull distribution.

III. Time Effects

Complicated histories of structural loads usually are mixtures of proof loads, cyclic loads and sustained loads. A simplified typical load history for pressure vessels is shown in Fig. 2. During cyclic and sustained loading above a certain threshold value, subcritical flaw growth is expected. In this paper, it is assumed that the time relationships of these subcritical flaw growths are representable by deterministic relations which are determined experimentally from tests on preflawed coupons by varying both the flaw size and the applied stress. The changes of the following ratios for the cyclic and the sustained cases, respectively, can be written in the general form,[9]

$$S_{cj}/R_{cj} = W(n) \tag{6}$$

$$S_{sj}/R_{sj} = U(t) \tag{7}$$

in which S_{cj} and S_{sj} are, respectively, the applied stresses due to cyclic and sustained loading. R_{cj} and R_{sj} are the corresponding strengths of the jth volume element before the application of S_{cj} and S_{sj}, and n indicates the number of cycles and t the time to failure. $W(n)$ and $U(t)$ are monotonically decreasing functions of their arguments.

The function $U(t)$ usually has a characteristic shape which can be approximated by an exponential decay to the threshold value b as follows:

$$U(t) = b + re^{-t/\tau} \tag{8}$$

where r is a parameter and τ a characteristic time. t is the time to failure in logscale.

In this paper the justifiable assumption, Ref. 9, is made that subcritical flaw growth, i.e., the time dependent deterioration of structural strength, under a short duration proof load test is negligible. Considering then the loading history shown in Fig. 2, as well as the weakest link hypothesis the deteriorated strength of the structure after application of the proof load S_0, and then after application of n loading cycles with amplitude S_c can be derived from Eqs. (1) and (6) in the following form:

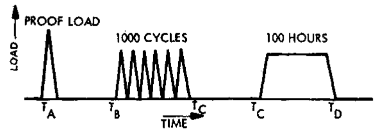

Fig. 2 Typical load history for pressure vessels.

$$\begin{aligned} R(n) &= S_c/W[W^{-1}(S_c/R_0) - n] && \text{for } W^{-1}(S_c/R_0) - n > 0 \\ &= 0 && \text{O.W.} \end{aligned} \tag{9}$$

where R_0 is the structural strength after proof load application. Similarly, using Eqs. (1, 7 and 9), the deteriorated strength after application of the sustained load with amplitude S_s for a time period T following application of n loading cycles with amplitude S_c is

$$\begin{aligned} R(T) &= S_s/U\{U^{-1}[S_s/R(n)] - T\} && \text{for } 1 > S_s/R(n) > b \\ & && U^{-1}[S_s/R(n)] - T > 0 \\ &= R(n) && \text{for } b \geq S_s/R(n) \\ &= 0 && \text{O.W.} \end{aligned} \tag{10}$$

In Eqs. (9) and (10), $W^{-1}(x)$ and $U^{-1}(y)$ are the inverse functions of $W(x)$ and $U(y)$ in Eqs. (6) and (7), which represent nothing but the number of cycles to failure associated with $S_c/R_0 = x$ and the time to failure associated with $S_s/R(n) = y$, respectively. Similar expressions as those in Eqs. (9) and (10) can be derived if the loading sequence of S_c and S_s is interchanged.

IV. Probability of Failure

Those structures which have passed the proof load test are to be used in actual mission applications. The original strength distribution for these vessels has been changed by the proof load test. This must be accounted for in the reliability evaluation. If R_0 is the structural strength after the application of the proof load S_0, then the structural strength distribution is given by the conditional probability

$$F_{R_0}(x) = P[R \leq x | R > S_0] = \frac{F_R(x) - F_R(S_0)}{1 - F_R(S_0)} \quad \text{for } x \geq S_0 \tag{11}$$

Use of Eq. (4) gives

$$F_{R_0}(x) = 1 - \exp\left\{-\frac{1}{v}\int_V \left[\left(\frac{x(\phi_1 + \phi_2) - x_\mu}{x_0}\right)^k - \left(\frac{S_0(\phi_1 + \phi_2) - x_\mu}{x_0}\right)^k\right] dv\right\} \tag{12}$$
$$\text{for } x(\phi_1 + \phi_2) \geq S_0(\phi_1 + \phi_2) \geq x_\mu$$

Eq. (12) is valid when $x(\phi_1 + \phi_2)$ is proportional to $S_0(\phi_1 + \phi_2)$. If this proportionality does not hold, the strength distribution, after application of S_0, is given by

$$F_{R_0}(x) = 1 - \exp\left\{-\frac{1}{v}\int_V \left[\left(\frac{x(\phi_1 + \phi_2) - x_\mu}{x_0}\right)^k - \left(\frac{S_0(\bar{\phi}_1 + \bar{\phi}_2) - x_\mu}{x_0}\right)^k\right] dv\right\} \tag{13}$$
$$\text{for } x(\phi_1 + \phi_2) \geq S_0(\bar{\phi}_1 + \bar{\phi}_2) \geq x_\mu$$

For the sake of simplifying the algebra, Eq. (12) is used in the following without essential loss of generality. The probability of structural failure, p_0, due to the proof load S_0 follows then from Eq. (3) as

$$p_0 = F_R(S_0) = 1 - \exp\left\{-\frac{1}{v}\int_V \left[\frac{S_0(\phi_1 + \phi_2) - x_\mu}{x_0}\right]^k dv\right\} \quad (14)$$

Let p_c be the probability of structural failure due to n cycles of the cyclic load S_c after S_0 has been applied, and let p_{cs} be the probability of structural failure due to the sustained load S_s for a period of time T, given that the structure has survived S_c. Then, it follows from Eqs. (6) and (7) that

$$p_c = P[S_c/R_0 \geq W(n)] = \int_0^\infty F_{R_0}\left(\frac{x}{W(n)}\right) f_{S_c}(x)\, dx \quad (15)$$

and

$$p_{cs} = P[S_s/R(n) \geq U(T)|R(n) > 0] = \int_0^\infty F_{R(n)}\left(\frac{x}{U(T)}\, |R(n) > 0\right) f_{S_s}(x) dx \quad (16)$$

In Eqs. (15) and (16) $f_{S_c}(x)$ and $f_{S_s}(x)$ are, respectively, the probability density functions of S_c and S_s, and $F_{R_0}(.)$ is given by Eq. (12).

The conditional distribution function $F_{R(n)}(x/U(T)|R(n) > 0)$ of $R(n)$ given $R(n) > 0$ can be obtained from Eq. (9) as follows:

$$F_{R(n)}[x|R(n) > 0] = P\left[\frac{S_c}{W[W^{-1}(S_c/R_0) - n]} \leq x\right] = \int_0^\infty F_{R_0}\left(\frac{y}{W[W^{-1}(y/x) + n]}\right) f_{S_c}(y) dy \quad (17)$$

Substitution of Eq. (17) into Eq. (16) yields

$$p_{cs} = \int_0^\infty \int_0^\infty F_{R_0}\left(\frac{y}{W\{W^{-1}[yU(T)/x] + n\}}\right) \times f_{S_c}(y)\, f_{S_s}(c) dy dx \quad (18)$$

in which it should be realized that $F_{R_0}(x) = 0$ for $x \leq S_0$.

The probability of structural failure due to the application of sustained loading S_s for a period of time T after passing the proof load test, i.e., without applying the cyclic loading S_c, is

$$p_s = P[S_s/R_0 \geq U(T)] = \int_0^\infty F_{R_0}\left(\frac{x}{U(T)}\right) f_{S_s}(x) dx \quad (19)$$

The probability of failure p_{sc} due to n cycles of S_c given that the structure has survived S_s for a period T after the application of S_0, can be obtained in a similar fashion as the probability of failure p_{cs}.

V. Optimization

Obtaining the best possible performance, or the least possible cost, or the least possible weight, etc., is an integral part of every structural design. The optimization task is to find the values of the controllable parameters, subject to the various constraints, that make a desired objective function an extremum. In this paper, the objective function to be optimized is the structural weight or the statistically expected cost, i.e., the mean cost due to coupon testing, proof testing, and mission degradation. The structural weight is expressible in terms of the physical parameters such as density and structural dimensions, and the cost items are expressible in terms of the proof load test levels, as well as the physical parameters. It is the objective of the present optimization process to determine those proof load test levels and physical parameters which yield minimum expected cost, or which yield minimum weight subject to an expected cost constraint.

Coupon testing has, as its prime purpose, the characterization of the statistical strength properties of the structural material. The efforts and costs which must be expended in order to establish, with sufficient confidence, the material strength distributions for one or more environmental conditions can be substantial. In particular, if this information is required at the tail end of the distributions, the number of coupon tests and the associated cost soon becomes intolerably high. Thus, in the over-all cost picture, the required expenditures for material characterization should be taken into account.

As can be seen from Eq. (12), the truncated structural strength distribution, $F_{R_0}(x)$, after the application of the proof load, is zero for strength values less than S_0. Thus the lower tail of the original strength distribution $F_R(x)$ does not give any contribution to the probability of failure. The upper tail of the truncated strength distribution contributes to the probability of failure depending on its relative interaction with the upper tail of the load distribution. Since the interaction between load and strength distribution is of a general form, as shown in Fig. 1, the upper tail contribution to the probability of failure diminishes very quickly with increasing distance away from the mean strength. Thus, the greatest contribution to the probability of failure during service stems from the region close to the proof load. Since the determination of the probability of failure with certain confidence requires knowledge of the distribution functions of the load and of the truncated strength R_0, it can be inferred that the required cost of coupon tests to enable the determination of the probability of failure with certain confidence is strongly dependent on the proof load level. This cost will be called coupon testing cost or material characterization cost. If different structural subsystems are used with differing materials, the total coupon testing cost is the sum of the characterization costs for each material.

The expected cost due to proof testing is the statistically expected cost of structural testing in which one structure after another is tested at a certain proof load level, until a structure is obtained which passes the applied proof load. This cost includes the cost of the structures after their completion plus the actual cost of proof testing. This structural qualification cost is also strongly dependent on the proof load test level as can be easily seen from Eq. (14). It should be recalled that the term structure refers here to structural subsystems such as struts, pressure vessels, etc., and that the structural system, such as a spacecraft structure, may consist of more than one subsystem. If a number of structural subsystems require qualification, then the total proof testing cost is the sum of the statistically expected costs due to proof testing for each subsystem.

From a structural utilization point of view, it is not only important to consider the costs of coupon testing and proof testing, but also the cost which will be incurred if the structure fails during the time of its use. In space applications, this cost may range from cost of total mission loss to negligibly small cost, depending on whether structural failure occurs at the beginning of a mission or after the mission objectives have been fulfilled, and depending on whether structural failure causes complete destruction or only some mission degradation. The statistically expected value of this cost, which will be called mission degradation cost, is the product of the actual cost of mission degradation times the probability of occurrence of this degradation, which is the probability of structural failure.

From the aforementioned remarks, the total statistically expected cost, EC, for n structural subsystems is the sum of the statistically expected costs, EC_i, for each ith structural subsystem and can be written as

$$EC = \sum_{i=1}^{n} EC_i \quad (20a)$$

$$EC_i = C_i(\epsilon_i) + q_i(\epsilon_i)C_{0i} + p_{fi}(\epsilon_i,\nu_i)C_f \tag{20b}$$

where the three terms on the right hand side of Eq. (20b) represent coupon testing cost, proof testing cost and mission degradation cost, respectively, of the ith subsystem. The proof test level ϵ_i for the ith structural subsystem is the ratio of the proof load S_{0i} to the mean structural strength $\bar{R}_i$; C_i is the coupon testing cost for the ith structural subsystem; q_i is the expected number of the ith structural subsystem failing before the one surviving the proof load is obtained; C_{0i} is the cost of loosing one of the ith structural subsystem during proof load; C_f is the actual cost of mission degradation; p_{fi} is the probability of structural failure of the ith structural subsystem during the mission. The approximation of the summation sign for the mission degradation cost in Eq. (20) is on the conservative side.[2,3] It follows, from the developments in the previous section, that p_{fi} is not only a function of ϵ_i but also of the central safety factor, $\nu_i = \bar{R}_i/\bar{S}_i$, which is the ratio of mean strength $\bar{R}_i$ to mean load $\bar{S}_i$, or of some other central measure of location. It should be noted that ν_i is numerically different from the conventional safety factor, which is usually based on percentiles of R_i and S_i, but plays, in principle, the same role.

The equations in Sec. IV are valid for any ith structural subsystem. Since $S_{0i} = \epsilon_i\bar{R}_i$, the probability of failure p_{0i} of the ith structural subsystem due to the proof load S_{0i} given by Eq. (14) can be expressed in terms of ϵ_i. It can be shown that

$$q_i(\epsilon_i) = p_{0i}(\epsilon_i)/[1 - p_{0i}(\epsilon_i)] \tag{21}$$

which gives the functional dependence of q_i on ϵ_i.

The coupon testing cost requires some explanation. Note that it is the subsystem which is subjected to proof testing thus truncating the original subsystem strength R_i into R_{0i}. Since the distribution of R_i and hence the distribution of R_{0i} are derived from the distribution of the coupon strength R_{ci} for the ith subsystem, i.e., Eq. (2), and since only the strength distribution function associated with the strength value $R_i \geq S_{0i}$, i.e., R_{0i} is needed in evaluating the probability of failure, it is necessary to establish the coupon strength distribution, with certain statistical confidence, only for those coupon strength values $R_{ci} > S_{ci}$ with S_{ci} being the value associated with the truncation point S_{0i} (or proof load) of the subsystem strength. Let $\bar{\epsilon}_i$ be the ratio of S_{ci} to the mean coupon strength $\bar{R}_{ci}$ for the ith subsystem, i.e. $\bar{\epsilon}_i = S_{ci}/\bar{R}_{ci}$. The functional dependence of C_i on $\bar{\epsilon}_i$ should be such that C_i is increasing with increasing absolute difference between $\bar{\epsilon}_i$ and 1, i.e., with $|\bar{\epsilon}_i - 1|$. Thus $\bar{\epsilon}_i$ should be expressible as a function of ϵ_i so that C_i in Eq. (20) can be written as a function of ϵ_i. In this paper, the following assumptions are made: 1) In the Weibull distribution, the parameter x_u is equal to zero, and 2) in Eq. (4), for each ith structural subsystem the expression $(\phi_1 + \phi_2)$ is independent of the space coordinates. Although the first assumption is not particularly restrictive, the second assumption implies a homogenious stress field within each structural subsystem. With these two assumptions, and using Eqs. (3) and (5), it can be easily shown that $\bar{\epsilon}_i = \epsilon_i$.

It is now assumed that the coupon testing cost for the ith structural subsystem can be approximated by an expression of the following form, Ref. 3;

$$C_i(\epsilon_i) = A_i + \delta_i B_i|\epsilon_i - 1|^{m_i} \tag{22}$$

where A_i is the minimum cost of coupon tests necessary for the determination of the mean value of coupon strength with certain confidence. B_i and m_i are constants, and δ_i is a constant which may take different values δ_i^+ and δ_i^- for $\epsilon_i > 1$ and $\epsilon_i < 1$, respectively.

If $\epsilon_i < 1$, the significant part of the truncated strength distribution for the evaluation of the probability of failure is located between the proof load level and the central portion of the strength distribution, whereas, if $\epsilon_i > 1$, the significant part lies beyond the central portion of the distribution. To establish the strength distribution with certain confidence, this suggests that a larger sample of coupons is required if $\epsilon_i > 1$ as compared to $\epsilon_i < 1$ for the same value $|\epsilon_i - 1|$.

Dividing Eq. (20) through by C_f and considering the previous remarks, the total relative expected cost $EC^* = EC/C_f$ and the relative expected cost for the ith subsystem $EC_i^* = EC_i/C_f$, becomes

$$EC^* = \sum_{i=1}^{n} EC_i^* \tag{23a}$$

$$EC_i^* = \alpha_i + \delta_i\beta_i|\epsilon_i - 1|^{m_i} + q_i(\epsilon_i)\gamma_i + p_{fi}(\epsilon_i,\nu_i) \tag{23b}$$

where $\alpha_i = A_i/C_f$, $\beta_i = B_i/C_f$ and $\gamma_i = C_{0i}/C_f$. Note that β_i and γ_i indicate the relative importance of the ith subsystem with respect to the actual cost of mission degradation if the ith subsystem fails and these values are the important parameters in the optimization process as will be shown later.

The optimization problem can now be stated as either to minimize the structural weight subject to a constraint on the relative expected cost given in Eq. (23), or to minimize the relative expected cost EC^*, subject to a constraint on the structural weight. Both approaches are essentially the same. In this paper, the optimization problem is stated as follows:

Minimize the total structural weight G subject to the maximum expected cost constraint $EC^* \leq EC_a^*$.

The objective function

$$G = \sum_{i=1}^{n} G_i$$

with G_i being the ith subsystem weight, can be written as a linear function of the design variables h_i, $i = 1,2,\ldots,n$, where h_i may, e.g., represent the cross-sectional area of the ith subsystem strut or the thickness of the ith subsystem pressure vessel, thus

$$G = \sum_{i=1}^{n} g_i h_i \tag{24}$$

where the g_i represent functions of physical and geometrical parameters of the ith subsystem.

It is emphasized that if the proof load test is not performed or is not considered, i.e., if all $q_i(\epsilon_i) = 0$, and if the material properties are well known to engineers so that coupon tests are not needed, i.e., if all $\alpha_i = \beta_i = 0$, then the maximum constraint EC_a^* becomes the maximum constraint of the probability of failure, and the problem reduces to the optimum design based on reliability constraint criteria which was discussed in the literature, e.g., Refs. 13–18.

Since there is only one constraint, Eq. (23), and the objective function is linear, in h_i, the constraint is always active, i.e., at optimum, the equality sign should hold in the constraint. Note that the central safety factor ν_i in Eq. (23) can be expressed as a function of h_j, $j = 1,2,\ldots,n$, i.e. $\nu_i = \nu_i(h_1,h_2,\ldots,h_n)$. The summation sign for the mission degradation cost in Eq. (20) assumes that the failure of each subsystem is statistically independent. Then with the method of Lagrangian multipliers, one can show that at optimum the following equations hold

$$\partial EC_i^*/\partial\epsilon_i = 0; \; i = 1,2,\ldots,n \tag{25a}$$

$$EC^* = EC_a^* \tag{25b}$$

Eq. (25) states that for an optimum structural weight, the proof load level ϵ_i, to be applied to the ith structural subsystem should also be optimum in the sense that corresponding to a given safety factor ν_i, the relative expected cost should be minimum at that level.

As for the optimization technique for statically indeterminate systems, or for cases where the failure of the subsystems is statistically dependent, gradient move methods can be employed. This has been discussed in the literature, e.g., Refs. 2 and 18.

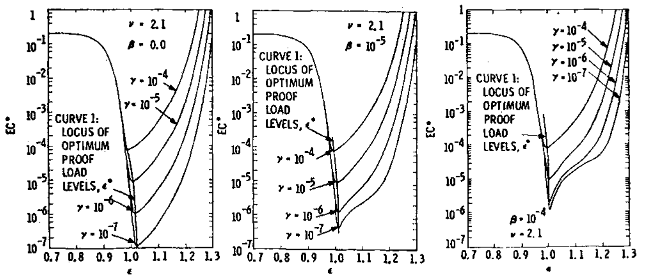

Fig. 3 Relative expected cost EC^* as a function of proof stress level ϵ; load distribution is normal with coefficient of variation 0.02 and strength distribution is truncated Weibull with coefficient of variation 0.10.

VI. Numerical Example

A 20-in.-diam spherical pressure vessel is designed to sustain after proof testing an internal pressure S_s for 360 hr. The weight of the vessel is to be minimized for an appropriate choice of proof load level ϵ (or S_0) and vessel wall thickness h in such a way that the total relative expected cost EC^* does not exceed a certain assigned value EC_a^* (constraint). The sustained pressure is assumed to have a Gaussian distribution with mean value $\bar{S}_s = 1000$ psi and 2% coefficient of variation. (Since only one subsystem is considered, the subscripts i are dropped.) It is assumed that the material is titanium Ti-61A-4V and that the coupon strength has a Weibull distribution with a mean value of $\bar{R}_c = 160{,}000$ psi and coefficient of variation of 10%, with a coupon size of 8-in. long, $\frac{1}{2}$-in. wide and $\frac{1}{4}$-in. thick. Based on the remarks of the last section, it is further assumed that in Eq. (22) $\delta^- = 1$ for $\epsilon < 1$ and $\delta^+ = 2$ for $\epsilon > 1$, and $m = 1$. The vessel is to be designed for room temperature at which the parametric values in Eq. (8) for $U(t)$ are $b = 0.5$, $r = 0.5$ and $\tau = 0.713$. The stress field is assumed so that $\phi_1 = \phi_2 = \phi =$ constant.

The Weibull distribution for the coupon strength is given in Eq. (2) from which, it follows that

$$\sigma_{R_c} = \{\Gamma(1 + 2/k) - \Gamma^2(1 + 1/k)\}^{1/2}/\Gamma(1 + 1/k) \quad (26)$$

and

$$\bar{R}_c = x_0\Gamma(1 + 1/k)/(V_c/v)^{1/k} \quad (27)$$

where σ_{R_c} is the coefficient of variation of the coupon strength equal to 0.1; and $\bar{R}_c$ is the mean coupon strength equal to 160,000 psi. Hence, the parameter k can be evaluated from Eq. (26), and then x_0 can be computed from Eq. (27).

Following the discussion in the previous sections one obtains:

$$\bar{R} = \frac{x_0}{2\phi(V/v)^{1/k}}\Gamma\left(1 + \frac{1}{k}\right); \; \phi = d/4h; \; \nu = \bar{R}/\bar{S}_s \quad (28)$$

$$h = \left\{\frac{\nu\bar{S}_s d(\pi d^2)^{1/k}}{2x_0\Gamma(1 + 1/k)}\right\}^{k/(k-1)} \quad (29)$$

$$q(\epsilon) = p_0(\epsilon)/[1 - p_0(\epsilon)] = \exp\{[\epsilon\Gamma(1 + 1/k)]^k\} - 1 \quad (30)$$

in which the diameter of the vessel, $d = 20$ in., the vessel material volume, $V = \pi d^2 h$, and the proof load level, $\epsilon = S_0/\bar{R}$ with S_0 being the proof load.

Using Eq. (19) and preceding equations, and making an appropriate transformation, the probability of vessel failure p_f due to S_s after the vessel passed the proof-test, can be put in the following form

$$P_f = \frac{1}{(2\pi)^{1/2}}\frac{\nu}{\sigma_s}\int_{\epsilon U(T)}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{x\nu - 1}{\sigma_s}\right)^2\right\} \times \left(1 - \exp\left\{-\left[\frac{x\Gamma(1 + 1/k)}{U(T)}\right]^k + [\epsilon\Gamma(1 + 1/k)]^k\right\}\right) dx \quad (31)$$

in which $\sigma_s = 0.02$ is the coefficient of variation of S_s, $U(T)$ is given in Eq. (8) with $T = 360$.

In this particular example, for one subsystem, Eq. (23) becomes

$$EC^* = \alpha + \delta\beta|\epsilon - 1| + q(\epsilon)\gamma + p_f(\epsilon,\nu) \quad (32)$$

The optimum values of ν and ϵ can now be determined from the following two equations

$$\partial EC^*/\partial\epsilon = 0; \quad EC^* = EC_a^* \quad (33)$$

The relative expected cost EC^* in Eq. (32) is plotted as a function of ϵ for different values of γ and β with $q(\epsilon)$ and $p_f(\epsilon,\nu)$ being given by Eqs. (30) and (31). These plots are shown in Fig. 3 for a particular value $\nu = 2.1$. The constant value α in Eq. (32) has been disregarded in these figures, since it has no effect on the optimization process. Including a nonzero value for α would give to the plots a corresponding shift parallel to the EC^* axis.

Those values of ϵ for which EC^* becomes minimum for a given ν are denoted by ϵ^*. The solution space of the optimum design, Eq. (33), can then be constructed by plotting the locuses of ϵ^*, such as Fig. 3, for different values of ν as shown in Fig. 4. This figure is the extended version of Fig. 3, and is referred to as the optimum design space.

The optimum design procedure can now be summarized as follows:

1) Construct the optimum design space, e.g., Fig. 4. 2) Read ν and ϵ^* from the optimum design space constructed in step 1. for specified constraint EC_a^* and given values of γ and β. 3) With the safety factor ν obtained in step 2, the optimum (minimum) thickness h of the vessel or the minimum weight G can be determined from Eq. 29.

Suppose the relative expected cost EC^* is to be minimized subject to the constraint on the vessel weight (or safety factor ν_a), the relationship of the first equation of Eq. (33) is still valid while the second should be replaced by $\nu = \nu_a$. Hence, the minimum relative expected cost design can be obtained either by plotting EC^* as a function of ϵ for given β,γ and constraint ν_a, such as Fig. 3, to find ϵ^* and minimum EC^*, or by reading ϵ^* and minimum EC^* directly from the optimum design space constructed previously, for a specified constraint ν_a and given values of γ and β. Numerical results for three specific cases are given in Table 1.

It should be noted that if the proof load test is not performed or not considered, i.e., $q(\epsilon) = 0$, and if the material property is well known so that the coupon tests are not needed, i.e., $\alpha = \beta = 0$, then, the maximum constraint EC_a^* on the relative expected cost becomes the maximum constraint on the probability of failure. The problem reduces then to the optimum design based on the reliability criteria. This design is termed as "Standard Optimum Design" in Table 1.

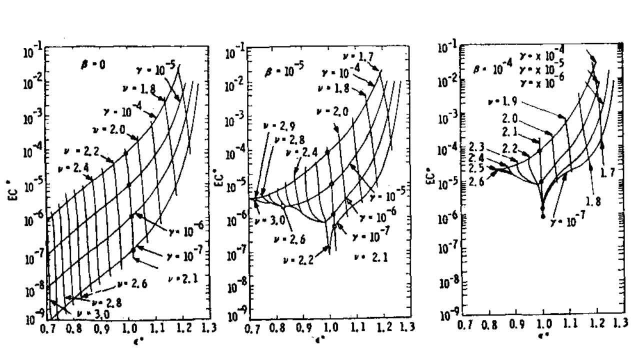

Fig. 4 Relative expected cost EC^* as a function of optimum proof stress level ϵ^*; load distribution is normal with coefficient of variation 0.02 and strength distribution is truncated Weibull with coefficient of variation 0.10.

Table 1 Optimum design of pressure vessel (h in in., S_0 in psi)

γ	10^{-7}	10^{-6}	10^{-5}	10^{-4}	Standard optimum design
		$\beta = 0.0$			
$EC_a^* = 10^{-6}$					
h, in. (thickness)	0.1869	0.2086	0.2472	0.300	0.6311
ϵ^*	1.11634	1.00845	0.8568	0.70928	0.0
S_0(psi)	2130	2128	2113	2089	0.0
ν	1.0982	2.1104	2.4668	2.9450	5.8300
p_f	0.7934×10^{-7}	0.4787×10^{-7}	0.4223×10^{-7}	0.706×10^{-7}	10^{-6}
$EC_a^* = 10^{-5}$					
h, in.	0.1746	0.1852	0.2065	0.2447	0.5134
ϵ^*	1.179	1.114	1.0091	0.850	0.0
S_0(psi)	2113	2108	2110	2088	0.0
ν	1.7925	1.8926	2.091	2.444	4.8238
p_f	0.174×10^{-5}	0.177×10^{-5}	0.472×10^{-6}	0.72×10^{-6}	10^{-5}
$EC_a^* = 10^{-4}$					
h, in.	0.1661	0.1727	0.1833	0.2042	0.4176
ϵ^*	1.2172	1.179	1.116	1.008	0.0
S_0(psi)	2090	2094	2091	2087	0.0
ν	1.713	1.775	1.874	2.07	3.99
p_f	0.374×10^{-4}	0.2115×10^{-4}	0.1242×10^{-4}	0.5731×10^{-5}	10^{-4}
		$\beta = 10^{-5}$			
$EC_a^* = 10^{-5}$					
h, in.	0.1765	0.1870	0.2066	0.2484	
ϵ^*	1.169	1.096	1.0086	0.8510	
S_0(psi)	2117	2093	2110	2089	
ν	1.81	1.91	2.092	2.477	
p_f	0.1042×10^{-5}	0.8236×10^{-6}	0.4639×10^{-6}	0.6941×10^{-6}	
$EC_a^* = 10^{-4}$					
h	0.1665	0.1728	0.1834	0.2042	
ϵ^*	1.2170	1.178	1.115	1.008	
S_0(psi)	2088	2092	2090	2087	
ν	1.716	1.776	1.876	2.07	
p_f	0.34×10^{-4}	0.1642×10^{-4}	0.1207×10^{-4}	0.573×10^{-5}	
		$\beta = 10^{-4}$			
$EC_a^* = 0.5 \times 10^{-4}$					
h	0.1734	0.1807	0.1920	0.2158	
ϵ^*	1.1865	1.138	1.073	0.9594	
S_0(psi)	2114	2104	2098	2090	
ν	1.781	1.850	1.960	2.178	
p_f	0.176×10^{-5}	0.349×10^{-5}	0.358×10^{-5}	0.253×10^{-5}	
$EC_a^* = 10^{-4}$					
h	0.1671	0.1744	0.185	0.2042	
ϵ^*	1.216	1.170	1.107	1.008	
S_0(psi)	2090	2094	2093	2087	
ν	1.7218	1.791	1.891	2.070	
p_f	0.1×10^{-4}	0.1277×10^{-4}	0.93×10^{-5}	0.574×10^{-5}	

VII. Discussion and Conclusion

Using Eq. (32), one obtains for different γ curves of the relative expected cost EC^* vs proof load level ϵ for different β and for different safety factors ν. Fig. 3 shows representative curves for four γ values, three β values and a particular safety factor $\nu = 2.1$. In Fig. 3, the relative expected cost EC^* changes very little for ϵ less than approximately 0.85. For other safety factors, EC^* behaves similarly, indicating that there is no pay-off in conducting proof testing below a certain value of ϵ. In all three cases, $\beta = 0$, $\beta = 10^{-5}$ and $\beta = 10^{-4}$, the optimum proof load levels are in the vicinity of $\epsilon = 1.0$ with a slight and expected tendency of the optimum proof load levels ϵ^* towards unity with increasing β. It is due to this fact that under reasonable relative expected cost constraints the optimum proof load level ϵ^* will fall with great likelihood within the range of two standard deviations around the mean value $\bar{R}$. From the designer's point of view, this is desirable, since in general, a considerably greater number of coupon tests is required for characterizing the truncated strength distribution with a certain level of statistical confidence if ϵ^* falls outside this region. It is noteworthy that for $\nu = 2.1$, EC^* is very sensitive to changes of ϵ in certain regions, such as $0.9 \lesssim \epsilon \lesssim 1.0$ and $\epsilon \gtrsim 1.1$, and for $\epsilon \gtrsim 0.97$, EC^* is also sensitive to γ. Similar statements hold also for safety factors different from $\nu = 2.1$.

In Figure 4, the relative expected cost EC^* is plotted as a function of the optimum proof load level ϵ^* for the same three values of β as in Fig. 3. Figure 4 is an extension of Fig. 3 in that the lines for $\nu = 2.1$ give the locus of optimum points indicated in Fig. 3, whereas the lines for the other values of ν reflect the locus of optimum points of similar curves as those in Fig. 3.

The first set of curves for $\beta = 0$ in Fig. 4 shows that when the coupon test is not needed, the relative expected cost EC^* can be made as small as desired simply by decreasing ϵ^* and increasing the safety factor ν, or the weight which, in this case, is proportional to ν. This result is a consequence of the fact that EC^* is here not a function of the cost of coupon tests. If the cost of coupon testing is considered, i.e., if $\beta \neq 0$ as for the second and third set of curves in Fig. 4, then the relative expected cost EC^* has a lower limit. This implies that in such cases a relative expected cost constraint less than this lower limit yields no feasible solution. It is evident from Fig. 4 that for $\beta \neq 0$ the absolute optimum proof load level is, in this case, in the vicinity of $\epsilon^* = 1.0$ for $\beta = 10^{-5}$ with $\nu = 2.2$ and for $\beta = 10^{-4}$ with $\nu = 2.1$. Additional details regarding the influence of the cost of coupon tests on the optimum design are given in Ref. 3.

Table 1 gives specific numerical results for different values of β and for specified relative expected cost constraints EC_a^*. It is particularly instructive to compare the results of the standard optimum design, $\epsilon^* = 0$, with those of the optimum design considering the proof load test, i.e., for $\gamma = 10^{-7}$ to 10^{-4}. Not only is considerable weight saving (weight is proportional to h) realized, but also a great reduction of the probability of failure p_f is obtained if the proof load level is considered as a design variable. The percentage of weight saving of the optimum design with proof load tests as compared to the design without proof load tests is much higher in this case than in the examples given in Refs. 2 and 3. This is because in the present case, the coefficient of variation of the strength R of the vessel ($\sigma_R = 10\%$) is higher than the coefficient of variation of loading ($\sigma_s = 2\%$) so that the probability of failure comes mainly from the lower portion of the strength distribution which is truncated by the proof load. In Refs. 2 and 3, low dispersion material ($\sigma_R = 5\%$) is used for high dispersion loading ($\sigma_s = 20\%$). The fact that the proof load test improves the statistical confidence of the reliability estimate has been discussed in Refs. 2 and 3.

Similar as in Refs. 2 and 3, the conclusion can be drawn here that the weight saving of the optimum design depends to a large degree on the parameter value γ. For low values of γ, one can afford to lose more vessels during proof load testing, i.e., higher values of ϵ can be applied, thus resulting in higher strength vessels and saving of structural weight. This follows from Fig. 4 and also from Table 1.

A general conclusion which can be drawn from the preceding remarks is that in proof testing structural subsystems it is to be expected that some of these subsystems are being lost. In fact, in cases where $\epsilon \gtrsim 1.0$, it should be expected that approximately half of these subsystems are destroyed during proof testing for the achievement of minimum expected cost EC^*. This is often incompatible with prevalent thinking during project applications, particularly if the subsystems are pressure vessels. It is often expected that no pressure vessel will be destroyed during proof testing and that pressure vessels are designed to fulfill that expectation. This implies that pressure vessels are designed for the proof load rather than the expected mission environment and are proof tested at a level ϵ which corresponds to the nearly horizontal portion of the curves for EC^* in Fig. 3. As stated previously, proof loads at such levels have no pay-off in terms of expected cost.

In the developments of this paper various simplifying assumptions were made which may be relaxed in a more extensive study. Nevertheless, it is felt that the results of this paper are representative and would not undergo major qualitative changes if these assumptions were relaxed, although quantitative changes would be expected.

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